DIVISORIAL CONTRACTIONS OF 3-FOLDS

S. A. KUDRYAVTSEV

Abstract. In this paper the three-dimensional divisorial contractions $f\colon Y \to (X \ni P)$ are classified provided that $\operatorname{Exc} f = E$ is an irreducible divisor, $f(E) = P$, Y has canonical singularities and $(X \ni P)$ is a toric terminal singularity.

Introduction

The divisorial contractions of 3-folds to the toric terminal singularities are studied in this paper. The main result is the following theorem.

Theorem. *Let $f\colon Y \to (X \ni P)$ be a divisorial contraction of 3-fold, where $\operatorname{Exc} f = E$ is an irreducible divisor and $f(E) = P$. Suppose that Y has canonical singularities and $(X \ni P)$ is a toric terminal singularity. Then either f is a toric morphism up to analytic isomorphism $(X \ni P) \cong (X \ni P)$, or f has a type $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$ and is described in section 2.*

Corollary. *In notations of theorem, suppose that f is not a toric morphism for any analytic isomorphism $(X \ni P) \cong (X \ni P)$. Then $(X \ni P)$ is either a non-singular point, or an ordinary double point.*

Corollary. [5], [1, Theorem 3.10], [2]. *In notations of theorem, suppose that Y has terminal singularities. Then f is a toric morphism up to analytic isomorphism $(X \ni P) \cong (X \ni P)$.*

The advantage of the proof given in this paper is that it does not use the classification of three-dimensional terminal singularities, log minimal model program and it can be generalized in the case of higher dimensions.

Let us remark that the results obtained have the very important applications to the rationality problem of algebraic varieties too. The theorem implies that a maximal singularity (in this class of singularities) is realized by some toric blow-up or has really very "rigid" structure.

The logarithmic version of this theorem is proved in [10].

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1. **Preliminary facts and results**

All varieties considered are defined over $\mathbb{C}$, the complex number field. The main definitions, notations and notions used in the paper are given in [6]. The germ of a variety X at a point P is denoted by $(X \ni P)$.

Proposition 1.1. [3, Lemma 4.4] *Let $f_i\colon Y_i \to X$ be two divisorial contractions of normal varieties, where $\operatorname{Exc} f_i = E_i$ are irreducible divisors and $-E_i$ are f_i-ample divisors. If E_1 and E_2 define the same discrete valuation of the function field $K(X)$ then the contractions f_1 and f_2 are isomorphic.*

Proposition 1.2. *Let $f_i\colon Y_i \to (X \ni P)$ be two divisorial contractions to a point P, where $\operatorname{Exc} f_i = E_i$ are irreducible divisors. Suppose that the varieties Y_i, X have log terminal singularities, E_1 and E_2 define the same discrete valuation of the function field $K(X)$, the divisor $-E_1$ is f_1-ample, the divisor $-E_2$ is not f_2-ample. Then there exists a flopping contraction (with respect to K_{Y_2}) $g\colon Y_2 \to Y_1$ and $f_2 = f_1 \circ g$.*

Proof. Let $K_{Y_2} = f_2^* K_X + aE_2$. If $a > 0$ then we put $L = -K_{Y_2}$. If $a \le 0$ then we put $L = -(K_{Y_2} + (-a+\varepsilon)E_2)$, where ε is a sufficiently small positive rational number. By base point free theorem [4, Remark 3.1.2] a linear system $|nL|$ is free over X for $n \gg 0$ and gives a contraction $g\colon Y_2 \to Y_2'$ over X. A curve C is exceptional for g if and only if $L \cdot C = E_2 \cdot C = K_{Y_2} \cdot C = 0$. Therefore h is a flopping contraction and $Y_2' = Y_1$ by proposition 1.1. □

Proposition 1.3. [8, Proposition 2.3] *Let $(X \ni P)$ be a three-dimensional toric terminal singularity. Then $X \cong (\mathbb{C}^3 \ni 0)/\mathbb{Z}_r(1,-q,q)$, where $(r,q) = 1$ or $X \cong \operatorname{Spec} \mathbb{C}[x_1,x_2,x_3,x_4]/(x_1x_2 + x_3x_4)$.*

Proof. If $(Z \ni P)$ is a $\mathbb{Q}$-factorial singularity then we have the first possibility by terminal lemma (for example, see [13, §1.6]). Assume that $(Z \ni P)$ is not a $\mathbb{Q}$-factorial singularity. Let $(Z' \ni P') \to (Z \ni P)$ be a canonical cover and let $\widetilde{Z} \to Z'$ be a $\mathbb{Q}$-factorialization. Since the variety $\widetilde{Z}$ has the $\mathbb{Q}$-factorial toric terminal singularities of index one only then $\widetilde{Z}$ is a smooth variety.

Let us prove that $\operatorname{Exc} f$ is an irreducible curve [14, Example 6]. Assume the converse. Since the divisor $K_{\widetilde{X}} + \widetilde{T}$ is log canonical, where $\widetilde{T}$ is a complement to the open toric orbit, then there exists an invariant irreducible (smooth) divisor $\widetilde{T}_1$ containing two intersecting curves from

$\operatorname{Exc} f$. It is impossible since these curves must be (-1) curves on $\widetilde{T}_1$. Hence $X' \cong \operatorname{Spec} \mathbb{C}[x_1, x_2, x_3, x_4]/(x_1x_2 + x_3x_4)$ [11, Exercise 14-2-10] and $X' = X$ by the classification of three-dimensional terminal singularities [12]. $\square$

Lemma 1.4. *Let $(\mathbb{C}^2_{x,y} \ni 0, D)$ be a log canonical but not purely log terminal pair, where $D = \sum \theta_i D_i = \theta_1\{x = 0\} + \theta_2\{y = 0\} + \sum_{i\geq 3} \theta_i\{f_i = 0\}$ and $x, y \nmid f_i$ for every $i \geq 3$. Put $d_x(f_i) = \min\{k \mid x^k \in f_i\}$ and $d_y(f_i) = \min\{k \mid y^k \in f_i\}$. Then two following statements hold.*

1) *There exists a decomposition $D = D_1' + D_2''$, where $D_1' = \theta_1 D_1 + \sum_{i\geq 3}^{j-1} \theta_i D_i + \theta_j' D_j$, $D_2' = \theta_2 D_2 + \theta_j'' D_j + \sum_{i>j} \theta_i D_i$ and $\theta_1 + \sum_{i\geq 3}^{j-1} \theta_i d_x(f_i) + \theta_j' d_x(f_j) = 1$.*
2) *We have $\theta_2 + \theta_j'' d_y(f_j) + \sum_{i>j} \theta_i d_y(f_i) \geq 1$.*

Remark 1.5. Before proving the lemma let us demonstrate its sense in this example. Consider a log canonical but not purely log terminal pair $(\mathbb{C}^2_{x,y} \ni 0, \frac{5}{6}\{x^2 + y^3 = 0\})$. We have $\frac{5}{6}\{x^2 + y^3 = 0\} = \frac{1}{2}\{x^2 + y^3 = 0\} + \frac{1}{3}\{x^2 + y^3 = 0\}$. Then $\frac{1}{2} d_x(x^2 + y^3) = \frac{1}{3} d_y(x^2 + y^3) = 1$.

Proof. If the decomposition does not exist then we put $D_1' = D$. Consider a deformation

$$D_t = \theta_1\{x = 0\} + \theta_2\{y = 0\} + \sum_{i\geq 3}^{j-1} \theta_i\{t^{-d_x(f_i)} \cdot f_i(tx, t^{d_x(f_i)+1}y) = 0\} +$$
$$\theta_j'\{t^{-d_x(f_j)} \cdot f_j(tx, t^{d_x(f_j)+1}y) = 0\} + \theta_j''\{t^{-d_y(f_j)} \cdot f_j(t^{d_y(f_j)+1}x, ty) = 0\} +$$
$$\sum_{i>j} \theta_i\{t^{-d_y(f_i)} \cdot f_i(t^{d_y(f_i)+1}x, ty) = 0\}.$$

Then

$$D_0 = \Big(\theta_1 + \sum_{i\geq 3}^{j-1} \theta_i d_x(f_i) + \theta_j' d_x(f_j)\Big)\{x = 0\} +$$
$$\Big(\theta_2 + \theta_j'' d_y(f_j) + \sum_{i>j} \theta_i d_y(f_i)\Big)\{y = 0\}.$$

If $(\mathbb{C}^2, D_0)$ is not a purely log terminal pair then the lemma is proved. Let $(\mathbb{C}^2, D_0)$ be a purely log terminal pair. Then $(\mathbb{C}^2, D_0 + \varepsilon C)$ is a purely log terminal pair for $0 < \varepsilon \ll 1$, where C is a general curve passing through 0. A contradiction with [7, Corollary 7.8]. $\square$

2. **Non-toric divisorial contractions**

In this section we construct the examples of non-toric divisorial contractions. The case of non-singular point and the case of ordinary double point are considered in items 2.1–2.5 and 2.6–2.8 respectively.

2.1. At first we consider the general construction of divisorial contractions. Then we prove its existence in special cases. After that our special cases are studied in detail.

Let $(X \ni P) = (\mathbb{C}^3 \ni 0)$ and $g\colon Z \to (X \ni P)$ be a weighted blow-up with weights $(\beta_1, \beta_2, \beta_3)$, $\operatorname{Exc} g = S$. Consider some irreducible curve Γ on S, which is not a toric subvariety of Z. Let $h\colon \widetilde{Y} \to Z$ be the blow-up of an ideal I_Γ, $\operatorname{Exc} h = \widetilde{E}$ and $\widetilde{Y} \to Y$ be the divisorial contraction of $\widetilde{S}$ to a point, where $\widetilde{S}$ is the proper transform of S. We obtain a divisorial contraction $f\colon Y \to (X \ni P)$, where $\operatorname{Exc} f = E$. In every example stated below the variety Y has the canonical but not terminal singularities and f is not a toric morphism.

Now we write our cases. Equality (1) gives the precise definition of a curve Γ.

Type $\mathbb{A}_n$) Put $(\beta_1, \beta_2, \beta_3) = (1, a_2d_1, a_3d_1)$, $S = \mathbb{P}(1, a_2, a_3)$, $\Gamma \sim \mathcal{O}_{\mathbb{P}(1,a_2,a_3)}(a_2 + a_3)$, $n = a_2d_1 + a_3d_1 - 1$.

Type $\mathbb{D}_{2k+2}$) Put $(\beta_1, \beta_2, \beta_3) = (2, 2k, 2k + 1)$, $S = \mathbb{P}(1, k, 2k + 1)$, $\Gamma \sim \mathcal{O}_{\mathbb{P}(1,k,2k+1)}(2k + 1)$, where $k \geq 1$.

Type $\mathbb{D}_{2k+1}$) Put $(\beta_1, \beta_2, \beta_3) = (2, 2k - 1, 2k)$, $S = \mathbb{P}(1, 2k - 1, k)$, $\Gamma \sim \mathcal{O}_{\mathbb{P}(1,2k-1,k)}(2k)$, where $k \geq 2$.

Type $\mathbb{E}_6$) Put $(\beta_1, \beta_2, \beta_3) = (3, 4, 6)$, $S = \mathbb{P}(1, 2, 1)$, $\Gamma \sim \mathcal{O}_{\mathbb{P}(1,2,1)}(2)$.

Type $\mathbb{E}_7$) Put $(\beta_1, \beta_2, \beta_3) = (4, 6, 9)$, $S = \mathbb{P}(2, 1, 3)$, $\Gamma \sim \mathcal{O}_{\mathbb{P}(2,1,3)}(3)$.

Type $\mathbb{E}_8$) Put $(\beta_1, \beta_2, \beta_3) = (6, 10, 15)$, $S = \mathbb{P}^2$, $\Gamma \sim \mathcal{O}_{\mathbb{P}^2}(1)$.

Consider a divisor $D = \{\varphi(x_1, x_2, x_3) + \psi(x_1, x_2, x_3) = 0\}$ on X, where the polynomial φ is quasihomogeneous with respect to weights $(\beta_1, \beta_2, \beta_3)$ and defines a corresponding Du Val singularity or $\varphi = x_1x_2^2 + x_3^2$ for types $\mathbb{D}_n$, $\mathbb{A}_3$ only. The polynomial ψ is general and has a large degree. We have

$$\Gamma = D_Z|_S = \widetilde{\{\varphi = 0\}}|_S, \tag{1}$$

where D_Z and $\widetilde{\{\varphi = 0\}}$ are the proper transforms of the divisors D and $\{\varphi = 0\}$. The pair (X, D) is canonical and $a(S, D) = a(\widetilde{E}, D) = 0$. Write $K_{\widetilde{Y}} + D_{\widetilde{Y}} = h^*g^*(K_X + D)$. By base point free theorem [4] the linear system $|nD_{\widetilde{Y}}|$ gives a required divisorial contraction $\widetilde{Y} \to Y$, where $n \gg 0$.

Note that the proper transform of a divisor D on Y is a general elephant for the divisorial contraction $Y \to (X \ni P)$.

Before considering the types $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_i$ explicitly let us prove the following lemma.

Lemma 2.2. *Let* $\widetilde{\Gamma} = \widetilde{S} \cap \widetilde{E}$. *Then*

$$(\widetilde{\Gamma}^2)_{\widetilde{E}} = \frac{\big(K_S + \mathrm{Diff}_S(0)\big) \cdot \Gamma}{a(S,0)+1} - (\Gamma^2)_S$$

and

1)

$$(\widetilde{\Gamma}^2)_{\widetilde{E}} = -\Big(\frac{1}{d_1}\frac{a_2+a_3}{a_2a_3} + \frac{(a_2+a_3)^2}{a_2a_3}\Big) \text{ for type } \mathbb{A}_n;$$

2)

$$(\widetilde{\Gamma}^2)_{\widetilde{E}} = -\Big(\frac{1}{2k} + \frac{2k+1}{k}\Big) \text{ for type } \mathbb{D}_{2k+2};$$

3)

$$(\widetilde{\Gamma}^2)_{\widetilde{E}} = -\Big(\frac{1}{2k-1} + \frac{4k}{2k-1}\Big) \text{ for type } \mathbb{D}_{2k+1};$$

4)

$$(\widetilde{\Gamma}^2)_{\widetilde{E}} = -\frac{13}{6}, -\frac{19}{12}, -\frac{31}{30} \text{ for types } \mathbb{E}_6,\ \mathbb{E}_7,\ \mathbb{E}_8 \text{ respectively.}$$

Proof. We have

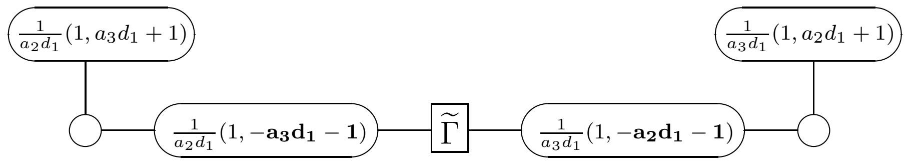
$$(\widetilde{\Gamma}^2)_{\widetilde{E}} = \widetilde{S} \cdot \widetilde{\Gamma} = S \cdot \Gamma - \widetilde{E} \cdot \widetilde{\Gamma} = S \cdot \Gamma - (\widetilde{\Gamma}^2)_{\widetilde{S}} = S \cdot \Gamma - (\Gamma^2)_S.$$

By adjunction theorem the lemma is proved. □

2.3. For every type $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_i$ the surface $\widetilde{E}$ is a conic bundle. By calculations (see item 2.5) every geometrical fiber is irreducible. The curve $\widetilde{\Gamma}$ is a minimal section by lemma 2.2. The surface E is obtained from $\widetilde{E}$ by a contraction of $\widetilde{\Gamma}$.

In the following diagrams we write a structure of $\widetilde{E}$. All singularities of $\widetilde{E}$ are written. The empty circle denotes a fiber (necessarily with singularities of $\widetilde{E}$). Every fiber f and $\widetilde{\Gamma}$ are the toric orbits near some cyclic singularity of type $\frac{1}{\gamma}(1,\mathbf{b})$ at a point $f \cap \widetilde{\Gamma}$. The bold weight b corresponds to $\widetilde{\Gamma}$. The singularities of $\widetilde{Y}$ outside $\widetilde{\Gamma}$ are written too (the variety Y has the same singularities).

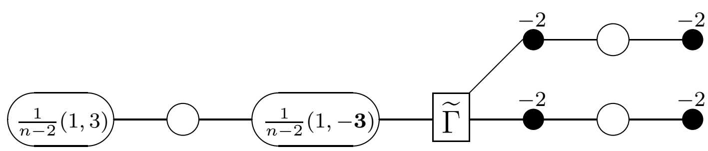
1) *Type* $\mathbb{A}_n$. The singularities of $\widetilde{Y}$ outside $\widetilde{\Gamma}$ are $\frac{1}{a_2d_1}(1, a_3d_1+1, -1)$ and $\frac{1}{a_3d_1}(1, a_2d_1+1, -1)$ respectively.



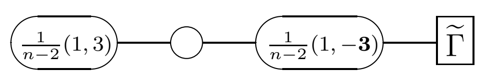
Note that the log surface $(E, \mathrm{Diff}_E(0))$ is toric.

2) *Type* $\mathbb{A}_3$, $a_2 = a_3 = 1$, $d_1 = 2$, $\varphi = x_1x_2^2 + x_3^2$. We have $\widetilde{\Gamma}^2 = -5$, $(\mathrm{Sing}\,\widetilde{Y}) \cap \widetilde{E} = f$ is a fiber, $\widetilde{Y} \cong \mathbb{A}^1 \times \frac{1}{2}(1,1)$ near $f \backslash \widetilde{\Gamma}$. The surface $\widetilde{E}$ is non-normal along f ($\mathrm{Sing}\,\widetilde{E} = f$) and the divisor $K_{\widetilde{Y}} + \widetilde{E}$ is log canonical but not purely log terminal.

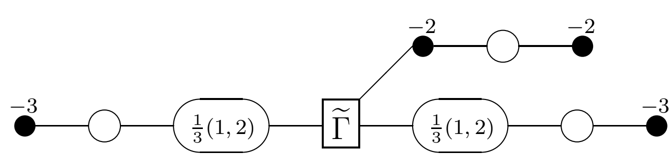
3) *Type* $\mathbb{D}_n$, $\varphi \neq x_1x_2^2 + x_3^2$. The singularities of $\widetilde{Y}$ outside $\widetilde{\Gamma}$ are $\frac{1}{n-2}(1,3,-1)$ and $2 \times \frac{1}{2}(1,1,1)$ respectively.



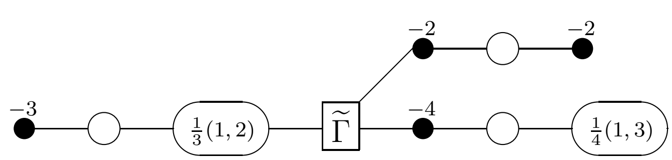
4) *Type* $\mathbb{D}_n$, $\varphi = x_1x_2^2 + x_3^2$. The singularities of $\widetilde{Y}$ outside $\widetilde{\Gamma}$ are $\frac{1}{n-2}(1,3,-1)$ and $\widetilde{Y} \cong \mathbb{A}^1 \times \frac{1}{2}(1,1)$ near $f_1 \backslash \widetilde{\Gamma}$, where f_1 is another fiber. The surface $\widetilde{E}$ is non-normal along f_1 and the divisor $K_{\widetilde{Y}} + \widetilde{E}$ is log canonical but not purely log terminal.



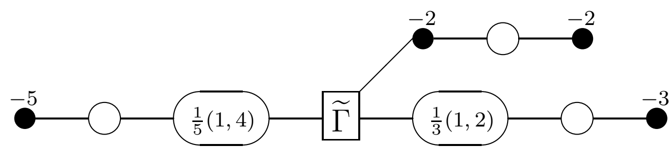
5) *Type* $\mathbb{E}_6$. The singularities of $\widetilde{Y}$ outside $\widetilde{\Gamma}$ are $2 \times \frac{1}{3}(1,1,-1)$ and $\frac{1}{2}(1,1,1)$ respectively.



6) *Type* $\mathbb{E}_7$. The singularities of $\widetilde{Y}$ outside $\widetilde{\Gamma}$ are $\frac{1}{3}(1,1,-1)$, $\frac{1}{4}(3,1,-1)$ and $\frac{1}{2}(1,1,1)$ respectively.



7) *Type* $\mathbb{E}_8$. The singularities of $\widetilde{Y}$ outside $\widetilde{\Gamma}$ are $\frac{1}{5}(1,1,-1)$, $\frac{1}{3}(1,1,-1)$ and $\frac{1}{2}(1,1,1)$ respectively.



Remark 2.4. In cases 1), 3), 4) it is possible that $\widetilde{Y}$ has the singularities along the fibers denoted by the circles. For example, in case 3) it is possible if and only if $3|(n-2)$.

2.5. Let us illustrate the calculations in cases 1) and 2). The other cases are analyzed similarly.

In case 1) the curve Γ passes through two different singular points P_1 and P_2 of Z. Consider the first singularity $\frac{1}{a_3d_1}(-1,-a_2d_1,1)$ at a point P_1. Since the curve Γ is a toric subvariety near P_1 we can use a toric geometry.

We refer the reader to [13] for the basics of a toric geometry. Let $N \cong \mathbb{Z}^n$ be a lattice of rank n and $M = \mathrm{Hom}(N,\mathbb{Z})$ its dual lattice. For a fan Δ in N the corresponding toric variety is denoted by $T_N(\Delta)$. For a k-dimensional cone $\sigma \in \Delta$ the closure of corresponding orbit is denoted by $V(\sigma)$. This is a closed subvariety of codimension k in $T_N(\Delta)$.

At the point P_1 we have $Z = T_N(\Delta)$, where $\Delta = \{\langle e_1,e_2,e_3\rangle,\ \text{their faces}\}$ and $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (1,a_2d_1,a_3d_1)$. Note that $S = V(\langle e_3\rangle)$, $\Gamma = V(\langle e_2,e_3\rangle)$. Then $\widetilde{Y} = T_N(\Delta')$, where

$$\Delta' = \{\langle e_4,e_1,e_2\rangle, \langle e_4,e_1,e_3\rangle,\ \text{their faces}\}$$

and $e_4 = e_2 + e_3 = (1, a_2d_1+1, a_3d_1)$. Note that $V(\langle e_4,e_1\rangle)$ is a fiber of $\widetilde{E}$ over P_1. By considering the cones $\langle e_4,e_1,e_2\rangle$ and $\langle e_4,e_1,e_3\rangle$ it is easy to prove the requirement. The case of a point P_2 is considered similarly.

In case 2) the curve Γ is tangent to $\operatorname{Sing} Z$. At the point of tangency we have

$$\big(Z \supset \Gamma\big) \cong \big(\mathbb{C}^3_{x_1,x_2,x_3} \supset \{x_1^2 + x_2^2 = x_3 = 0\}\big)/\mathbb{Z}_2(0,1,1).$$

To simplify calculations we define the variety Z in $\mathbb{C}^4_{y_1,y_2,y_3,y_4}$ by the equation $(y_2 - y_1^2)y_3 - y_4^2 = 0$ and the curve Γ by $y_2 = y_3 = y_4 = 0$. By considering usual affine pieces of a blow-up $\widetilde{Y} \to Z$ it is easy to prove the requirement.

2.6. Let $(X \ni P) = (x_1x_2 + x_3x_4 = 0 \subset (\mathbb{C}^4, 0))$ and a toric divisorial contraction $g\colon Z \to (X \ni P)$ is induced by the weighted blow-up of $\mathbb{C}^4$ with weights $(1, \beta_2, \beta_3, \beta_4)$, where $1 + \beta_2 = \beta_3 + \beta_4$, $\operatorname{Exc} g = S$. Consider an irreducible curve $\Gamma \sim \mathcal{O}_{\mathbb{P}(1,\beta_2,\beta_3,\beta_4)}(\beta_2)|_S$ on the surface $S = (x_1x_2 + x_3x_4 \subset \mathbb{P}(1, \beta_2, \beta_3, \beta_4))$, which is not a toric subvariety of Z. Let $\widetilde{Y} \to Z$ be the blow-up of an ideal I_Γ and $\widetilde{Y} \to Y$ be the divisorial contraction of a proper transform of S to point. We obtain a divisorial contraction $f\colon Y \to (X \ni P)$ (it is of type $\mathbb{A}_{\beta_2}$). The variety Y has the canonical but not terminal singularities and f is not a toric morphism. The existence of construction is proved similarly as in the case of non-singular point, where $D = \{x_1^{\beta_2} + x_2 + \ldots = 0\}|_X$ and $\Gamma = D_Z|_S$.

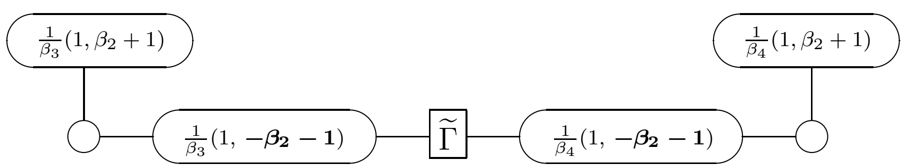
The following lemma is proved similarly to lemma 2.2.

Lemma 2.7. *We have*

$$(\widetilde{\Gamma}^2)_{\widetilde{E}} = -(\beta_2 + 1)\Big(\frac{1}{\beta_3} + \frac{1}{\beta_4}\Big).$$

Using the toric calculations as above-mentioned we obtain the answer.

2.8. The singularities of $\widetilde{Y}$ outside $\widetilde{\Gamma}$ are $\frac{1}{\beta_3}(1, \beta_2 + 1, -1)$ and $\frac{1}{\beta_4}(1, \beta_2 + 1, -1)$ respectively. Note that $\widetilde{Y}$ can be also singular along the fibers denoted by circles.



The log surface $(E, \operatorname{Diff}_E(0))$ is toric.

Remark 2.9. Note that the divisorial contractions $Y \to (X \ni P)$ constructed in this section are purely log terminal blow-ups except cases 2), 4). See the paper [10] about the classification of purely log terminal blow-ups of three-dimensional terminal toric singularities.

3. **Main theorem**

Theorem 3.1. *Let $f\colon Y \to (X \ni P)$ be a divisorial contraction of 3-fold, where $\operatorname{Exc} f = E$ is an irreducible divisor and $f(E) = P$. Suppose that Y has canonical singularities and $(X \ni P)$ is a toric terminal singularity. Then either f is a toric morphism up to analytic isomorphism $(X \ni P) \cong (X \ni P)$, or f has a type $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$ and is described in section 2.*

Corollary 3.2. *In notations of theorem* 3.1*, suppose that f is not a toric morphism for any analytic isomorphism $(X \ni P) \cong (X \ni P)$. Then $(X \ni P)$ is either a non-singular point, or an ordinary double point.*

Corollary 3.3. [5]*,* [1, Theorem 3.10]*,* [2]*. In notations of theorem* 3.1*, suppose that Y has terminal singularities. Then f is a toric morphism up to analytic isomorphism $(X \ni P) \cong (X \ni P)$.*

Proof. Let f be not a toric morphism for any analytic isomorphism $(X \ni P) \cong (X \ni P)$. Then there exists a toric divisorial contraction $g\colon Z \to X$ with the following properties:

I) $\operatorname{Exc} g = S$ is an irreducible divisor;

II) Γ is not a toric subvariety of Z for any analytic isomorphism $(X \ni P) \cong (X \ni P)$, where Γ is the center of a divisor E on Z;

III) by proposition 1.2 we may assume that $-S$ is g-ample divisor.

Let $D_Y \in |-nK_Y|$ be a general divisor for $n \gg 0$. Then the pair $(X, \frac{1}{n}D)$ is canonical and $a(E, \frac{1}{n}D)$=0, where $D = f(D_Y)$.

Lemma 3.4. *Let D_Z be the proper transform of a divisor D on Z. Then $a(E, S + \frac{1}{n}D_Z) \le -1$, in particular $\Gamma \subset \operatorname{LCS}(S, \operatorname{Diff}_S(\frac{1}{n}D_Z))$.*

Proof. Write $K_Z + \frac{1}{n}D_Z = g^*(K_X + \frac{1}{n}D) + \alpha S$, where $\alpha \ge 0$. If S is a Cartier divisor at a generic point of Γ then

$$a\Big(E, S + \frac{1}{n}D_Z\Big) \le a\Big(E, -\alpha S + \frac{1}{n}D_Z\Big) - 1 = -1.$$

Therefore we may assume that Γ is a point in generic position lying in the set $\operatorname{Supp}(\operatorname{Diff}_S(0))$. Since the pair (X, S) is purely log terminal then we have

$$\big(Z \supset S\big) \cong \big(\mathbb{C}^3_{x,y,z} \supset \{x = 0\}\big)/\mathbb{Z}_r(q, 1, 0)$$

in the small neighborhood of a point Γ by [6, Proposition 16.6]. Let $\theta\colon Z' \to Z$ be the blow-up along a curve $\{x = y = 0\}$ with weights $(\frac{q}{r}, \frac{1}{r})$ and $T = \operatorname{Exc}\theta$. Since

$$0 \le a\Big(T, -\alpha S + \frac{1}{n}D_Z\Big) = \frac{q+1-r}{r} + \frac{\alpha q}{r} - \operatorname{mult}\Big(\frac{1}{n}D_Z\Big),$$

then

$$a\Big(E, S + \frac{1}{n}D_Z\Big) = -(\alpha+1)\operatorname{mult}_\Gamma S \le -\left(\frac{1 - \frac{1}{r}}{\frac{q}{r}}\right)\frac{q+1}{r}$$

$$= -\Big(1 - \frac{1}{r}\Big)\Big(1 + \frac{1}{q}\Big) \le -1.$$

□

According to proposition 1.3 let us consider three cases in items 3.5, 3.9 and 3.10 respectively.

3.5. Let $(X \ni P)$ be a non-singular point. Then g is the weighted blow-up with weights $(\beta_1, \beta_2, \beta_3) = (a_1 d_2 d_3, a_2 d_1 d_3, a_3 d_1 d_2)$, $d_i = (\beta_j, \beta_k)$, where $j, k \neq i$. We have

$$\Big(S, \mathrm{Diff}_S(0)\Big) = \Big(\mathbb{P}_{x_1,x_2,x_3}(a_1, a_2, a_3), \sum_{i=1}^{3} \frac{d_i - 1}{d_i}\{x_i = 0\}\Big).$$

Lemma 3.6. *Let Γ be a (irreducible) curve. Then $a\big(S, \frac{1}{n}D\big) = 0$ and every irreducible component of D_Z contains Γ.*

Proof. Let $a\big(S, \frac{1}{n}D\big) > 0$. Then the multiplicity of a divisor $\frac{1}{n}D_Z$ along a curve Γ is more then 1. Let the curve Γ be defined by equation $\sum b_l x_1^{l_1} x_2^{l_2} x_3^{l_3} = 0$ in $\mathbb{P}(a_1, a_2, a_3)$. Consider an irreducible component D_1 of a divisor D. If the proper transform of D_1 on Z contains Γ then D_1 is given by equation

$$\varphi^k + \psi = \Big(\sum b_l x_1^{d_1 l_1} x_2^{d_2 l_2} x_3^{d_3 l_3}\Big)^k + \psi(x_1, x_2, x_3) = 0.$$

Since D_Y is a general divisor then the polynomial ψ is general and has a large degree by the construction of $Y \to (X \ni P)$. Hence $a\big(M, \frac{1}{n}D\big) < 0$, where M is an exceptional divisor obtained by the blow-up of a maximal ideal m_P. A contradiction. The second statement is proved similarly. $\square$

Corollary 3.7. *Let Γ be a curve. Then we have one of the following cases (up to change of variables), which are considered in section 2 :*
$\mathbb{A}_n$) $(\beta_1, \beta_2, \beta_3) = (1, a_2 d_1, a_3 d_1)$, $\Gamma \sim \mathcal{O}_{\mathbb{P}(1,a_2,a_3)}(a_2 + a_3)$;
$\mathbb{D}_{2k+2}$) $(\beta_1, \beta_2, \beta_3) = (2, 2k, 2k+1)$, $\Gamma \sim \mathcal{O}_{\mathbb{P}(1,k,2k+1)}(2k+1)$, $k \geq 1$;
$\mathbb{D}_{2k+1}$) $(\beta_1, \beta_2, \beta_3) = (2, 2k-1, 2k)$, $\Gamma \sim \mathcal{O}_{\mathbb{P}(1,2k-1,k)}(2k)$, $k \geq 2$;
$\mathbb{E}_6$) $(\beta_1, \beta_2, \beta_3) = (3, 4, 6)$, $\Gamma \sim \mathcal{O}_{\mathbb{P}(1,2,1)}(2)$;
$\mathbb{E}_7$) $(\beta_1, \beta_2, \beta_3) = (4, 6, 9)$, $\Gamma \sim \mathcal{O}_{\mathbb{P}(2,1,3)}(3)$;
$\mathbb{E}_8$) $(\beta_1, \beta_2, \beta_3) = (6, 10, 15)$, $\Gamma \sim \mathcal{O}_{\mathbb{P}^2}(1)$.

Proof. In notation of lemma 3.6 the divisor D is given by equation

$$\varphi^n + \psi = \Big(\sum b_l x_1^{d_1 l_1} x_2^{d_2 l_2} x_3^{d_3 l_3}\Big)^n + \psi(x_1, x_2, x_3) = 0.$$

We have three conditions:
A) the pair $\big(X, \frac{1}{n}D\big)$ is canonical;
B) φ is a quasihomogeneous polynomial with weights $(\beta_1, \beta_2, \beta_3)$;
C) $a\big(S, \frac{1}{n}D\big) = \sum \beta_i - 1 - \sum d_i l_i \beta_i = 0$.

It follows easily that φ is a polynomial defining Du Val singularity or $\varphi = x_3^2 + x_2^2 x_1$. $\square$

Lemma 3.8. *The set Γ is not a point.*

Proof. It is enough to prove that the divisor $K_S + \mathrm{Diff}_S(\frac{1}{n}D_Z)$ is nef. By lemma 3.4 the divisor $K_S + \mathrm{Diff}_S(\frac{1}{n}D_Z)$ is not purely log terminal. Let us consider a most difficult case, when the point Γ lies on the toric subvariety of a surface $S = \mathbb{P}(a_1, a_2, a_3)$. Another case is considered similarly. We can assume without loss of generality that $\Gamma = (0 : 1 : 1)$. Let the curves $\{x_1 = 0\}$ and $\{x_2^{a_3} - x_3^{a_2} = 0\}$ be the local coordinates of a point Γ. Then the divisor, which is composed by irreducible components of $\mathrm{Diff}_S(\frac{1}{n}D_Z)$ passing through a point Γ, is the sum of quasihomogeneous polynomials

$$\theta_1\{x_1 = 0\} + \theta_2\{x_2^{a_3} - x_3^{a_2} = 0\} + \sum_{i\geq 3}\theta_i\{(x_2^{a_3} - x_3^{a_2})^{k_i} + \ldots + x_1^{l_i} = 0\},$$

where $a_2, a_3 \geq 2$. By lemma 1.4 we have

$$\theta_1 a_1 + \sum_{i\geq 2}\theta_i k_i a_2 a_3 \geq a_1 + a_2 a_3.$$

Since the divisor $-(K_S + \mathrm{Diff}_S(\frac{1}{n}D_Z))$ is ample then $a_2 + a_3 - a_2 a_3 > 0$. Hence $a_2 = 1$ or $a_3 = 1$. Let $a_2 = 1$, $a_3 \geq 2$. Condition II) implies that $d_3 \geq 2$ and

$$a_2 + a_3 - a_2 a_3 - \frac{d_3 - 1}{d_3} a_3 = 1 - \frac{d_3 - 1}{d_3} a_3 < 0.$$

Hence the divisor $K_S + \mathrm{Diff}_S(\frac{1}{n}D_Z)$ is ample. Let $a_2 = a_3 = 1$. Then by the same argument we prove that $d_2, d_3 \geq 2$ and the divisor $K_S + \mathrm{Diff}_S(\frac{1}{n}D_Z)$ is ample. $\square$

The case of $(X \ni P)$ being a non-singular point is proved completely.

3.9. Let $(X \ni P) \cong (\mathbb{C}^3 \ni 0)/\mathbb{Z}_r(1, r-q, q)$. Let Γ be a curve. Write $D = \{h = 0\}/\mathbb{Z}_r(1, r-q, q)$. The proof of lemma 3.6 implies that $\frac{1}{n}\mathrm{mult}_P\{h = 0\} \geq 2$. Hence $a(M, \frac{1}{n}D) < 0$, where M is the exceptional divisor of the weighted blow-up with weights $(\frac{1}{r}, \frac{r-q}{r}, \frac{q}{r})$. A contradiction.

Let Γ be a point. The proof of lemma 3.8 implies that the divisor $K_S + \mathrm{Diff}_S(\frac{1}{n}D_Z)$ is nef, a contradiction.

3.10. Let $(X \ni P) \cong \mathrm{Spec}\,\mathbb{C}[x_1, x_2, x_3, x_4]/(x_1x_2 + x_3x_4)$. It is clear that the morphism g is induced by the weighted blow-up of $\mathbb{C}^4$ with weights $(\beta_1, \beta_2, \beta_3, \beta_4)$, where $\beta_1 + \beta_2 = \beta_3 + \beta_4$ and $(X \ni P) = (x_1x_2 + x_3x_4 = 0 \subset (\mathbb{C}^4, 0))$. In particular, $(\beta_i, \beta_j, \beta_k) = 1$ for all mutually distinct i, j, k. Put $(\beta_1, \beta_2, \beta_3, \beta_4) =$

$(a_1d_{23}d_{24}, a_2d_{13}d_{14}, a_3d_{14}d_{24}, a_4d_{13}d_{23})$, where $d_{ij} = (\beta_k, \beta_l)$ $(i, j, k, l$ are mutually distinct numbers). Then

$$\Big(S, \mathrm{Diff}_S(0)\Big) = \Big(\mathbb{P}_{x_1,x_2,x_3,x_4}(\beta_1, \beta_2, \beta_3, \beta_4), \sum_{i<j} \frac{d_{ij}-1}{d_{ij}}\{x_i = x_j = 0\}\Big).$$

The details of calculation can be found in the paper [9]. Note that $\rho(S) = 2$ since a complement to the open toric orbit of S consists of four irreducible curves C_i. Every Weil divisor of S is linearly equivalent to some sum of C_i with integer coefficients. Therefore, if Γ is a point then the divisor $K_S + \mathrm{Diff}_S(\frac{1}{n}D_Z)$ is nef for the same reason as in lemma 3.8, a contradiction.

Lemma 3.11. *Let Γ be a curve then $(\beta_1, \beta_2, \beta_3, \beta_4) = (1, a_2d_{13}d_{14}, a_3d_{14}, a_4d_{13})$ and $\Gamma \sim \mathcal{O}_{\mathbb{P}(1,\beta_2,\beta_3,\beta_4)}(\beta_2)|_S$ (up to change of variables). This case is considered in section 2.*

Proof. Every $\mathbb{Q}$-Cartier Weil divisor on X is Cartier divisor. Therefore the divisor D is given by $h(x_1, x_2, x_3, x_4) = 0$. In the same way as in the case of non-singular point we can prove that $\frac{1}{n}\,\mathrm{mult}_P(h) = 1$ and $a(S, \frac{1}{n}D) = 0$. In particular, $h = (x_i + \ldots)^n + \psi$ for some i. This directly implies that $\beta_j = 1$ for some $j \neq i$. □

□

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Department of Algebra, Faculty of Mathematics, Moscow State University, GSP-2 119992, Vorob'evy Gory, Moscow, Russia

E-mail address: `kudryav@mech.math.msu.su`